Accounting for model errors in iterative ensemble smoothers

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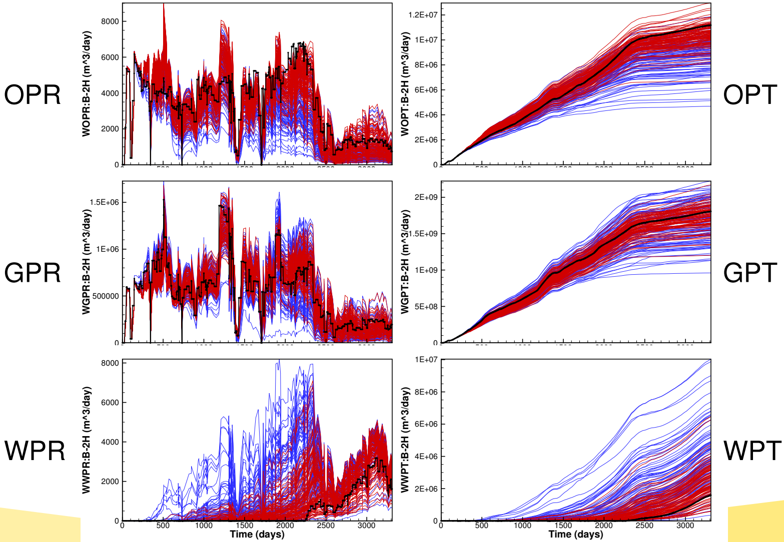
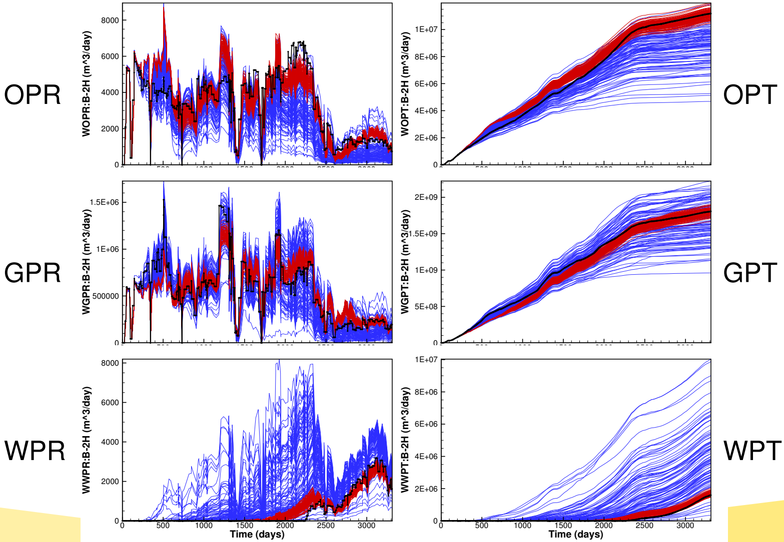
Example: ES



OPR
WOPR:B-2H (m^3/day)
OPT
WOPT:B-2H (m^3/day)
GPR
WGPR:B-2H (m^3/day)
GPT
WGPT:B-2H (m^3/day)
WPR
WWPR:B-2H (m^3/day)
WPT
WWPT:B-2H (m^3/day)
Time (days)
Time (days)

Example: An iterative ES



Inverse problem

Find $\boldsymbol{x} \in \Re^n$ given a model

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$$

given measurements $\boldsymbol{d} \in \Re^m$ of $\boldsymbol{y} \in \Re^m$

$$\boldsymbol{d} = \boldsymbol{y} + \boldsymbol{e}$$

- Standard History-Matching problem in oil-reservoir models.

Data-assimilation problem

Find $\boldsymbol{x}^{\mathrm{a}}_{i+1} \in \Re^n$ given a model prediction

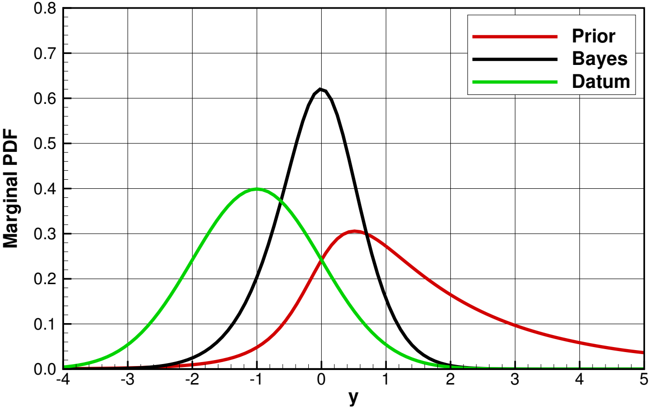
$$\boldsymbol{x}_{i+1} = \boldsymbol{g}(\boldsymbol{x}_i)$$

and measurements $\boldsymbol{d} \in \Re^m$ of $\boldsymbol{x}_{i+1} \in \Re^n$

$$\boldsymbol{d} = \boldsymbol{h}(\boldsymbol{x}_{i+1}) + \boldsymbol{e}$$

- Standard update step in sequential data assimilation

Bayesian update



Ensemble filter update

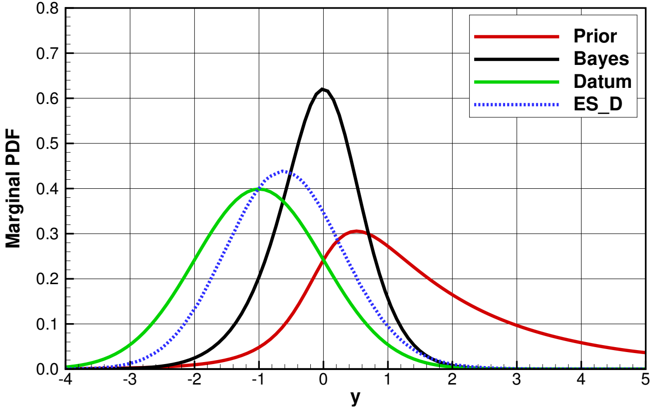
Prior

$$\boldsymbol{y}_j^{\mathrm{f}} = \boldsymbol{g}(\boldsymbol{x}_j^{\mathrm{f}})$$

Direct update of $\boldsymbol{y}$

$$\boldsymbol{y}_j^{\mathrm{a}} = \boldsymbol{y}_j^{\mathrm{f}} + \overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} \big(\overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} + \boldsymbol{C}_{dd}\big)^{-1} \Big(\boldsymbol{d}_j - \boldsymbol{y}_j^{\mathrm{f}}\Big)$$

EnKF (direct) update



Ensemble smoother update

Prior

$$\boldsymbol{y}_j^{\mathrm{f}} = \boldsymbol{g}(\boldsymbol{x}_j^{\mathrm{f}})$$

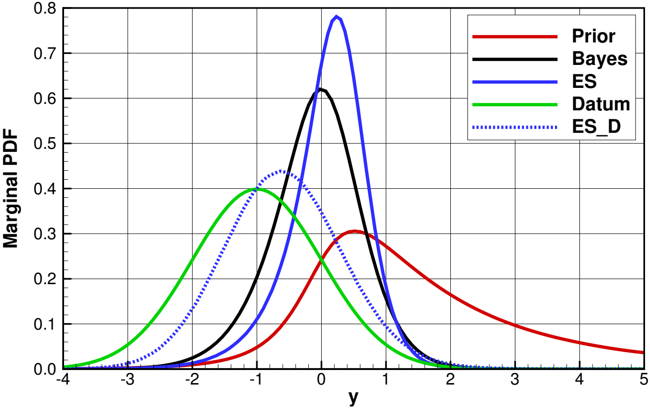
Smoother update of $\boldsymbol{x}$

$$\boldsymbol{x}_j^{\mathrm{a}} = \boldsymbol{x}_j^{\mathrm{f}} + \overline{\boldsymbol{C}}_{xy}^{\mathrm{f}} \left(\overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} + \boldsymbol{C}_{dd}\right)^{-1} \left(\boldsymbol{d}_j - \boldsymbol{y}_j^{\mathrm{f}}\right)$$

Indirect update of $\boldsymbol{y}$

$$\boldsymbol{y}_j^{\mathrm{a}} = \boldsymbol{g}(\boldsymbol{x}_j^{\mathrm{a}})$$

ES (indirect) update



Indirect update

Inverse problems: Measure $\boldsymbol{y}$ $\rightarrow$ Update $\boldsymbol{x}$ $\rightarrow$ predict $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$

Data assimilation: Measure $\boldsymbol{x}_{i+1}$ $\rightarrow$ Update $\boldsymbol{x}_i$ $\rightarrow$ predict $\boldsymbol{x}_{i+1} = \boldsymbol{g}(\boldsymbol{x}_i)$

- The indirect update allows for the use of iterative methods.

Nonlinearity

Sensitivity of smoothers to nonlinearity

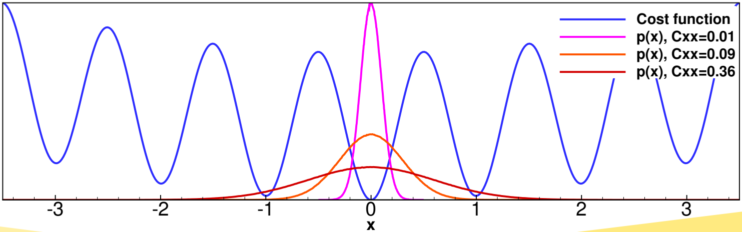
Model

$$y = 1 + \sin(\pi x)$$

Case 1: $x_j^{\mathrm{f}} \leftarrow \mathcal{N}(0.0, C_{xx} = 0.01)$ $\quad d_j \leftarrow \mathcal{N}(1.0, C_{dd} = 0.01)$

Case 2: $x_j^{\mathrm{f}} \leftarrow \mathcal{N}(0.0, C_{xx} = 0.09)$ $\quad d_j \leftarrow \mathcal{N}(1.0, C_{dd} = 0.01)$

Case 3: $x_j^{\mathrm{f}} \leftarrow \mathcal{N}(0.0, C_{xx} = 0.36)$ $\quad d_j \leftarrow \mathcal{N}(1.0, C_{dd} = 0.01)$



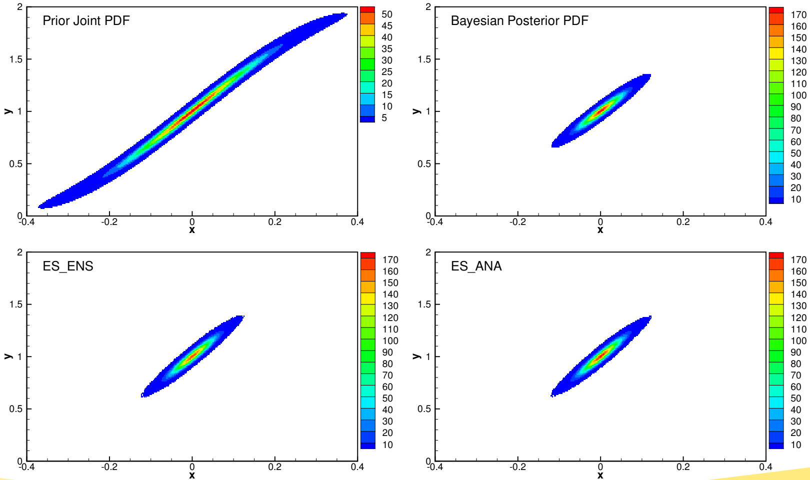
Ensemble vs analytic gradient

$$x_j^{\mathrm{a}} = x_j^{\mathrm{f}} + g'(x_j^{\mathrm{f}}) C_{xx} \left(g'(x_j^{\mathrm{f}}) C_{xx} g'(x_j^{\mathrm{f}}) + C_{dd} \right)^{-1} \left(d_j - g(x_j^{\mathrm{f}}) \right)$$

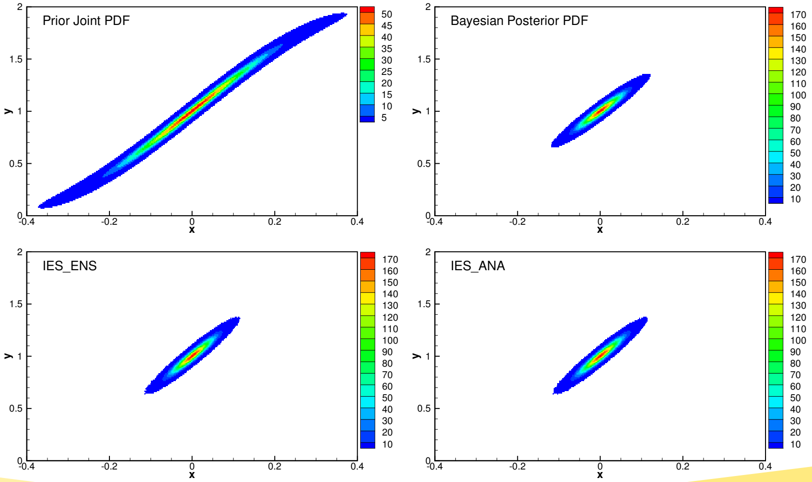
$$x_j^{\mathrm{a}} = x_j^{\mathrm{f}} + \overline{C}_{xy} (\overline{C}_{yy} + \overline{C}_{dd})^{-1} (d_j - g(x_j^{\mathrm{f}}))$$

$$y_j^{\mathrm{a}} = g(x_j^{\mathrm{a}})$$

Case 1: ES


Prior Joint PDF
Bayesian Posterior PDF
ES_ENS
ES_ANA
x
y
-0.4 -0.2 0 0.2 0.4
0 0.5 1 1.5 2
50 45 40 35 30 25 20 15 10 5
170 160 150 140 130 120 110 100 90 80 70 60 50 40 30 20 10

Case 1: IES


Prior Joint PDF
Bayesian Posterior PDF
IES_ENS
IES_ANA
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y
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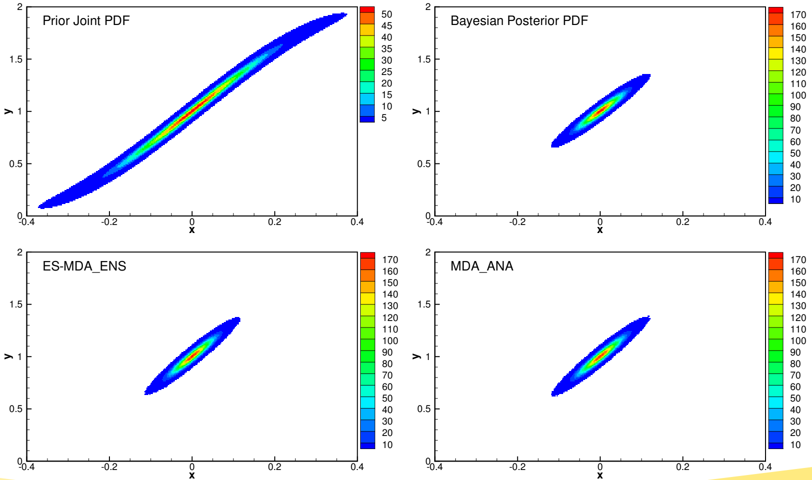
10

Case 1: ESMDA


Prior Joint PDF
Bayesian Posterior PDF
ES-MDA_ENS
MDA_ANA
x
y
-0.4
-0.2
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0.2
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1
1.5
2
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25
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15
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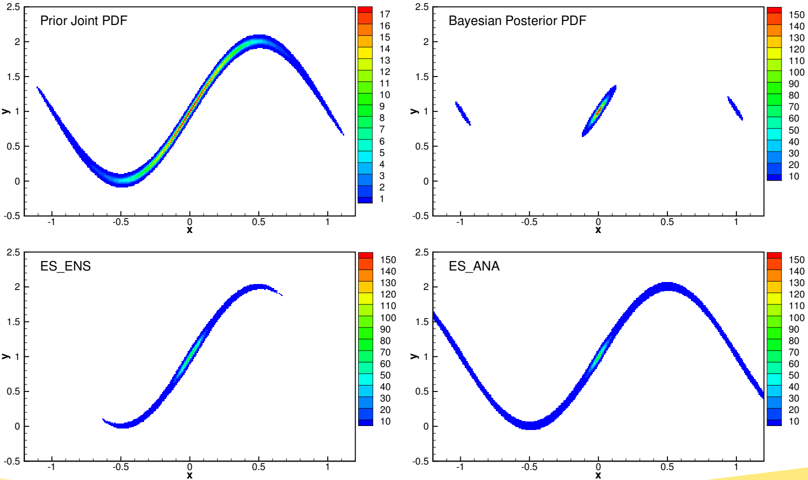
Case 2: ES


Prior Joint PDF
Bayesian Posterior PDF
ES_ENS
ES_ANA
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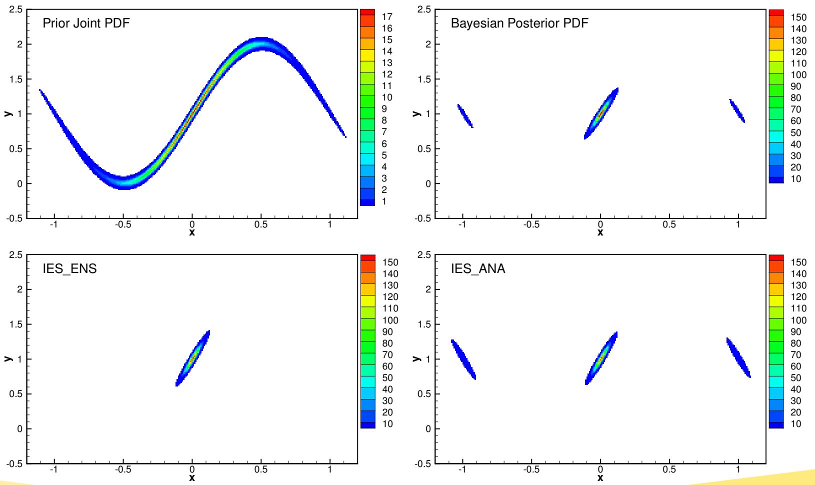
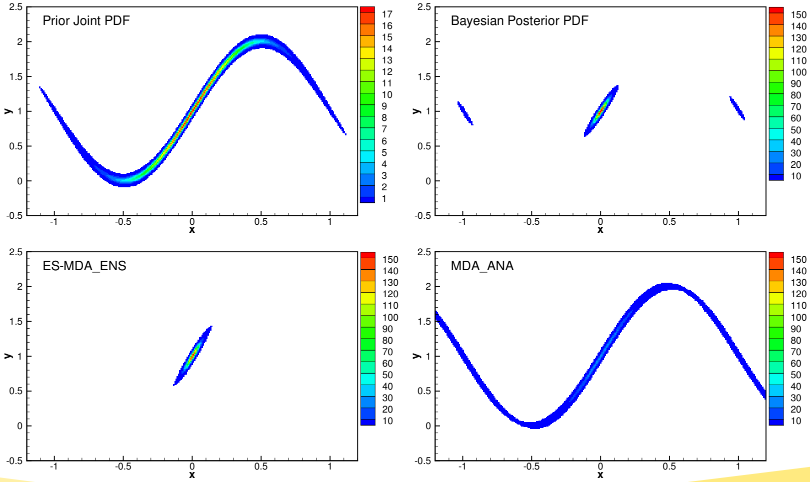
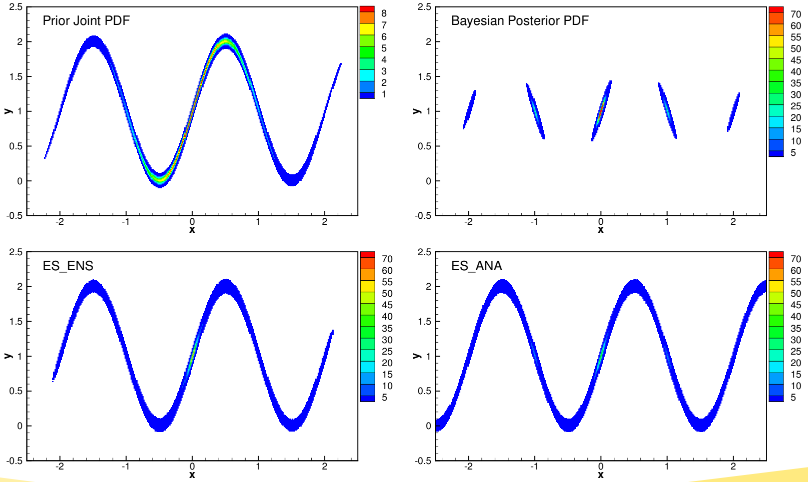
Case 2: IES



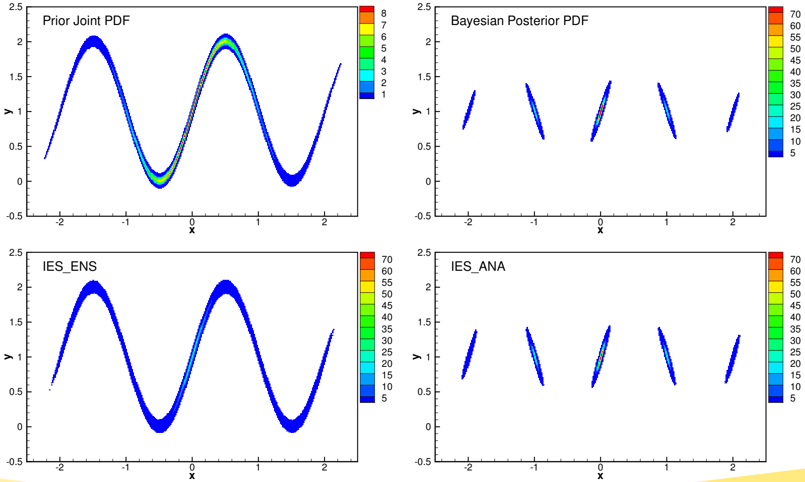
Case 2: ESMDA



Case 3: ES



Case 3: IES



Prior Joint PDF
Bayesian Posterior PDF
IES_ENS
IES_ANA
x
y
8
7
6
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3
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70
60
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35
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15
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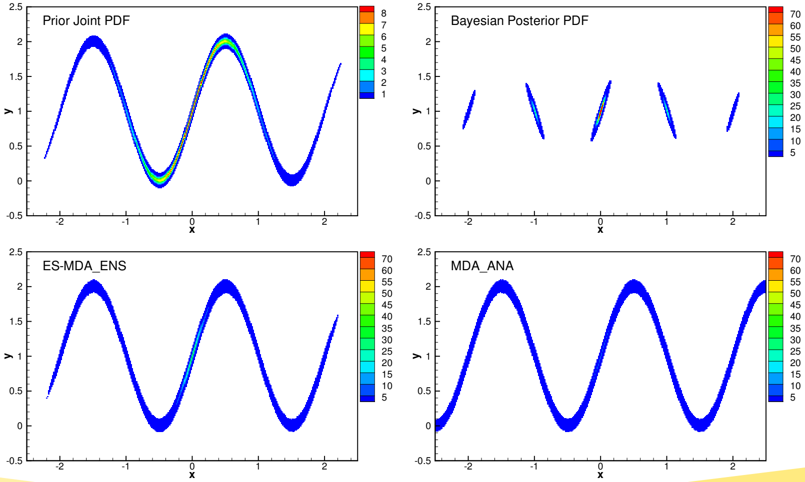
5
2.5
2
1.5
0.5
0
-0.5
-2
-1

Case 3: ESMDA



Model errors in iterative smoothers

Including model errors

Model

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{q})$$

Bayes

$$f(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{q} | \boldsymbol{d}) \propto f(\boldsymbol{d}|\boldsymbol{y}) \delta\big(\boldsymbol{y} - \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{q})\big) f(\boldsymbol{x}) f(\boldsymbol{q})$$

Marginal pdf

$$f(\boldsymbol{x}, \boldsymbol{q} | \boldsymbol{d}) \propto f\big(\boldsymbol{d} | \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{q})\big) f(\boldsymbol{x}) f(\boldsymbol{q})$$

Ensemble of cost functions

$$\begin{aligned}\mathcal{J}(\boldsymbol{x}_j, \boldsymbol{q}_j) = & \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right)^{\mathrm{T}} \boldsymbol{C}_{xx}^{-1} \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right) \\ & + \left(\boldsymbol{q}_j - \boldsymbol{q}_j^{\mathrm{f}}\right)^{\mathrm{T}} \boldsymbol{C}_{qq}^{-1} \left(\boldsymbol{q}_j - \boldsymbol{q}_j^{\mathrm{f}}\right) \\ & + \left(\boldsymbol{g}(\boldsymbol{x}_j, \boldsymbol{q}_j) - \boldsymbol{d}_j\right)^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}(\boldsymbol{x}_j, \boldsymbol{q}_j) - \boldsymbol{d}_j\right)\end{aligned}$$

ES including model errors

Prior

$$\boldsymbol{y}_j^{\mathrm{f}} = \boldsymbol{g}(\boldsymbol{x}_j^{\mathrm{f}}, \boldsymbol{q}_j^{\mathrm{f}})$$

Update

$$\boldsymbol{x}_j^{\mathrm{a}} = \boldsymbol{x}_j^{\mathrm{f}} + \overline{\boldsymbol{C}}_{xy}^{\mathrm{f}} (\overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} + \boldsymbol{C}_{dd})^{-1} \Big(\boldsymbol{d}_j - \boldsymbol{y}_j^{\mathrm{f}}\Big)$$

$$\boldsymbol{q}_j^{\mathrm{a}} = \boldsymbol{q}_j^{\mathrm{f}} + \overline{\boldsymbol{C}}_{qy}^{\mathrm{f}} (\overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} + \boldsymbol{C}_{dd})^{-1} \Big(\boldsymbol{d}_j - \boldsymbol{y}_j^{\mathrm{f}}\Big)$$

Indirect update

$$\boldsymbol{y}_j^{\mathrm{a}} = \boldsymbol{g}(\boldsymbol{x}_j^{\mathrm{a}}, \boldsymbol{q}_j^{\mathrm{a}})$$

or direct update

$$\boldsymbol{y}_j^{\mathrm{a}} = \boldsymbol{y}_j^{\mathrm{f}} + \overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} (\overline{\boldsymbol{C}}_{yy}^{\mathrm{f}} + \boldsymbol{C}_{dd})^{-1} \Big(\boldsymbol{d}_j - \boldsymbol{y}_j^{\mathrm{f}}\Big)$$

ESMDA including model errors

Cost function for $\boldsymbol{z}_j^{\mathrm{T}} = \left(\boldsymbol{x}_j^{\mathrm{T}}, \boldsymbol{q}_j^{\mathrm{T}}\right)$ at step i

$$\begin{aligned}\mathcal{J}(\boldsymbol{z}_{i+1}) &= \left(\boldsymbol{z}_{i+1} - \boldsymbol{z}_i\right)^{\mathrm{T}} \left(\boldsymbol{C}_{zz}^i\right)^{-1} \left(\boldsymbol{z}_{i+1} - \boldsymbol{z}_i\right) \\ &\quad + \left(\boldsymbol{g}(\boldsymbol{z}_{i+1}) - \boldsymbol{d} - \sqrt{\alpha_i}\boldsymbol{e}_i\right)^{\mathrm{T}} \left(\alpha_i \boldsymbol{C}_{dd}\right)^{-1} \left(\boldsymbol{g}(\boldsymbol{z}_{i+1}) - \boldsymbol{d} - \sqrt{\alpha_i}\boldsymbol{e}_i\right)\end{aligned}$$

where we must have

$$\sum_{i=1}^{N_{\text{mda}}} \frac{1}{\alpha_i} = 1.$$

$$\begin{aligned}\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i + \overline{\boldsymbol{C}}_{xy}^i \left(\overline{\boldsymbol{C}}_{yy}^i + \alpha_i \boldsymbol{C}_{dd}\right)^{-1} \left(\boldsymbol{d} + \sqrt{\alpha_i}\boldsymbol{e}_i - \boldsymbol{y}_i)\right) \\ \boldsymbol{q}_{i+1} &= \boldsymbol{q}_i + \overline{\boldsymbol{C}}_{qy}^i \left(\overline{\boldsymbol{C}}_{yy}^i + \alpha_i \boldsymbol{C}_{dd}\right)^{-1} \left(\boldsymbol{d} + \sqrt{\alpha_i}\boldsymbol{e}_i - \boldsymbol{y}_i)\right) \\ \boldsymbol{y}_{i+1} &= \boldsymbol{g}(\boldsymbol{x}_{i+1}, \boldsymbol{q}_{i+1})\end{aligned}$$

IES including model errors

Cost function for $\boldsymbol{z}_j^{\mathrm{T}} = \left(\boldsymbol{x}_j^{\mathrm{T}}, \boldsymbol{q}_j^{\mathrm{T}}\right)$

$$\mathcal{J}(\boldsymbol{z}_j) = \left(\boldsymbol{z}_j - \boldsymbol{z}_j^{\mathrm{f}}\right)^{\mathrm{T}} \boldsymbol{C}_{zz}^{-1} \left(\boldsymbol{z}_j - \boldsymbol{z}_j^{\mathrm{f}}\right) + \left(\boldsymbol{g}(\boldsymbol{z}_j) - \boldsymbol{d}_j\right)^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}(\boldsymbol{z}_j) - \boldsymbol{d}_j\right)$$

Iterative solution

$$\begin{aligned} \boldsymbol{z}_{j,i+1} &= \boldsymbol{z}_{j,i} + \overline{\boldsymbol{C}}_{zz}^{i} \overline{\boldsymbol{C}}_{zz}^{-1} (\boldsymbol{z}_{j,i} - \boldsymbol{z}_j^{\mathrm{f}}) \\ &\quad - \overline{\boldsymbol{C}}_{zy}^{i} \left(\overline{\boldsymbol{C}}_{yy}^{i} + \boldsymbol{C}_{dd}\right)^{-1} \left(\overline{\boldsymbol{C}}_{yz}^{i} \overline{\boldsymbol{C}}_{zz}^{-1} (\boldsymbol{z}_{j,i} - \boldsymbol{z}_j^{\mathrm{f}}) - \left(\boldsymbol{g}(\boldsymbol{z}_{j,i}) - \boldsymbol{d}_j\right)\right) \\ \boldsymbol{y}_{j,i+1} &= \boldsymbol{g}(\boldsymbol{z}_{j,i}) \end{aligned}$$

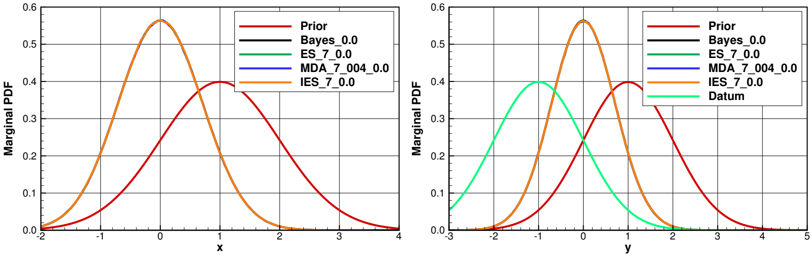
Scalar example

Model

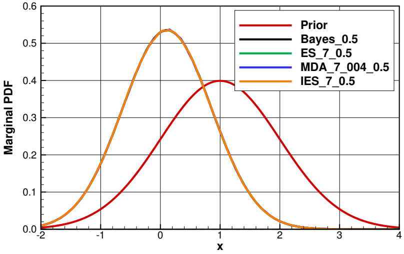
$$y = x(1 + \beta x^2) + \sqrt{C_{qq}} q$$

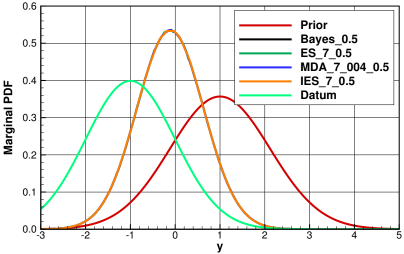
- Linear case: $\beta = 0$.
- Nonlinear case: $\beta = 0.2$.
- Prior ensemble for x: $N(1, 1)$.
- Likelihood for measurement of y: $N(-1, 1)$.

Linear model without model errors

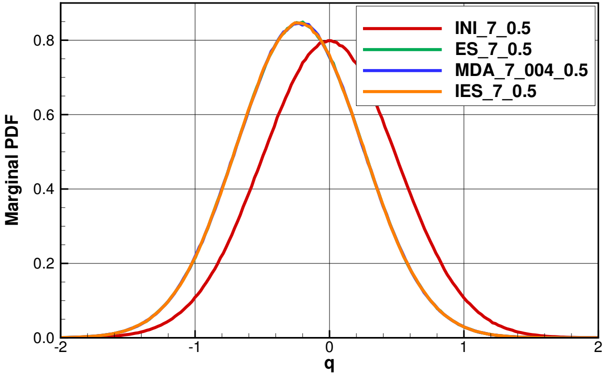


Linear model including model errors

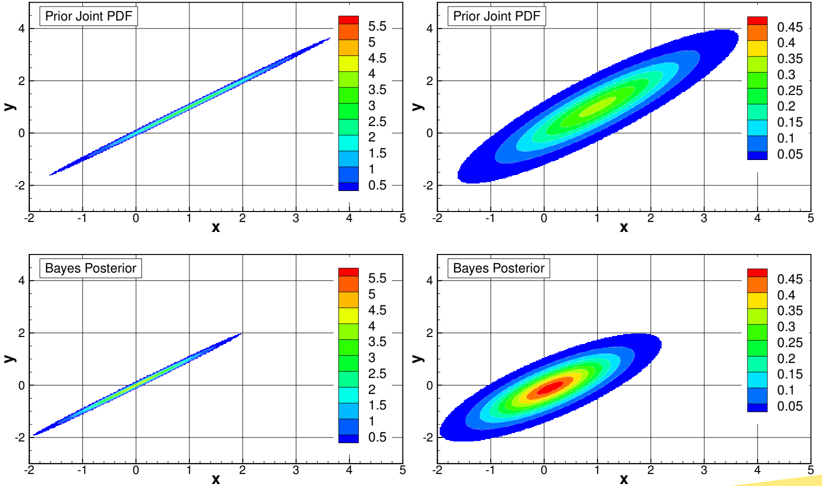




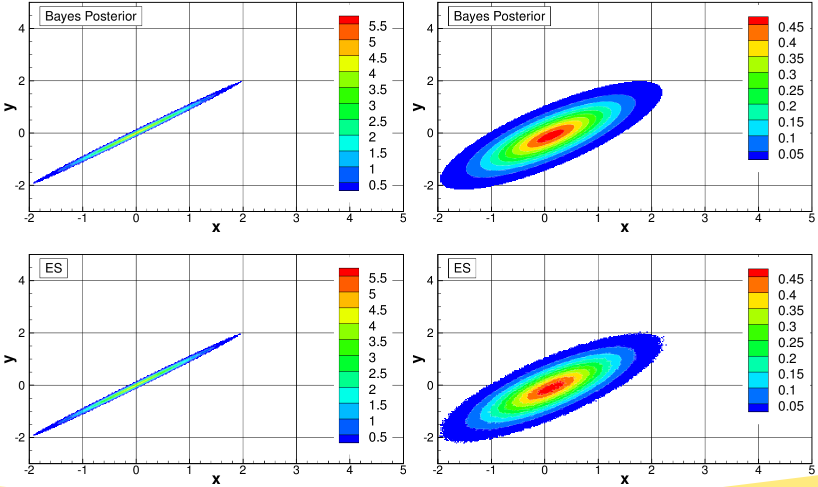
Pdfs for model error with linear model



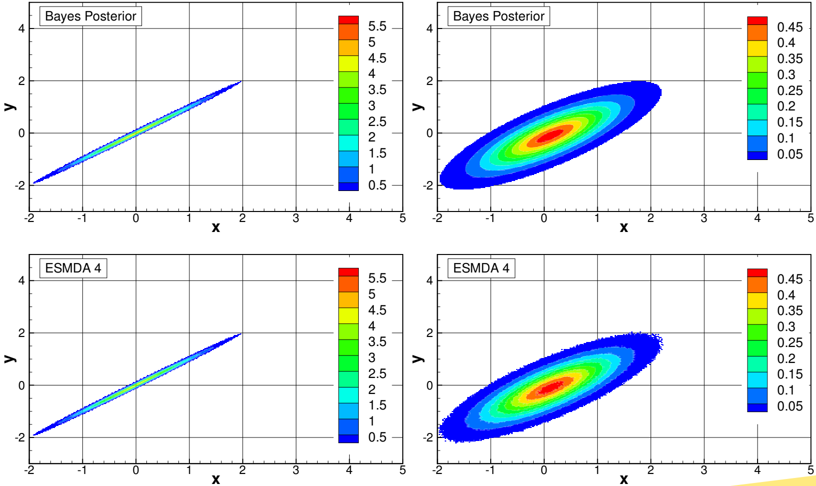
Joint pdfs linear model



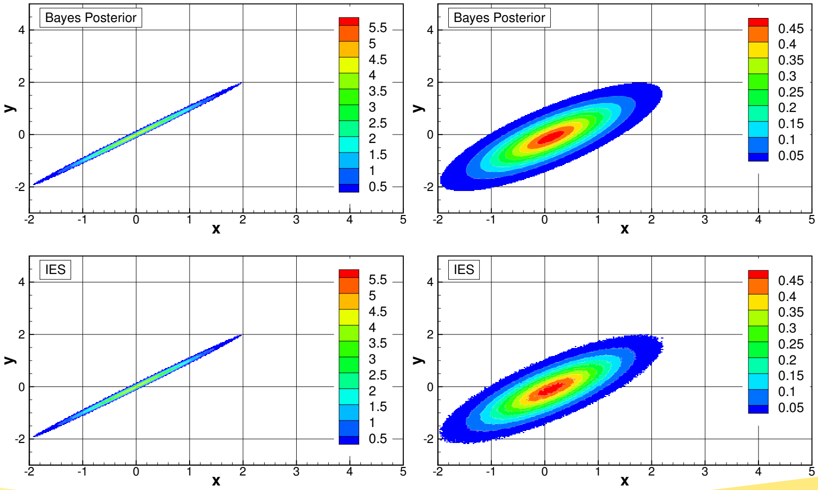
Joint pdfs linear model



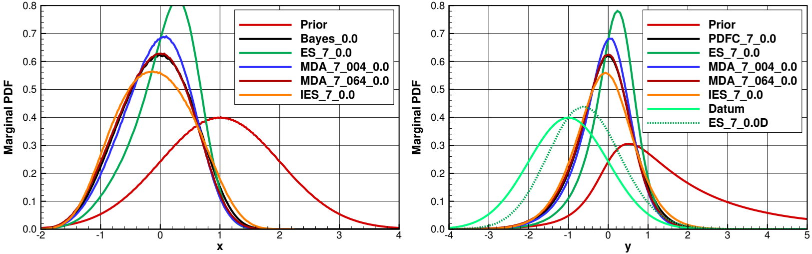
Joint pdfs linear model



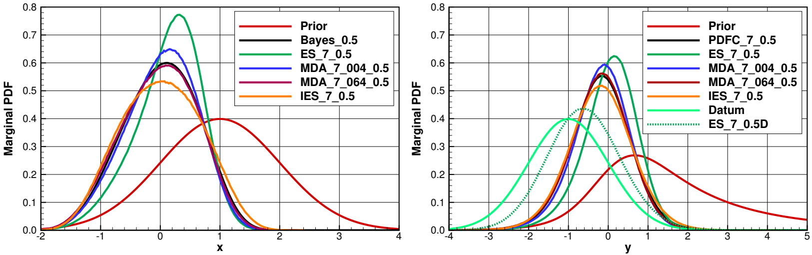
Joint pdfs linear model



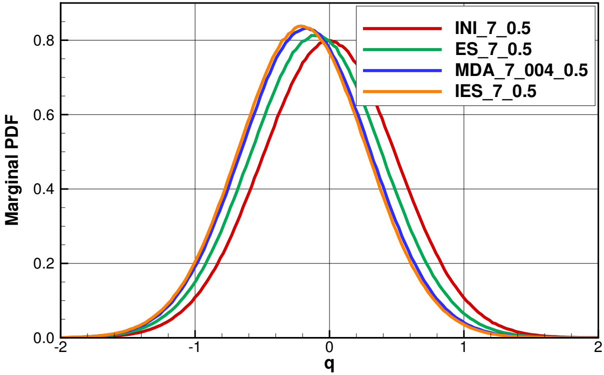
Nonlinear model without model errors



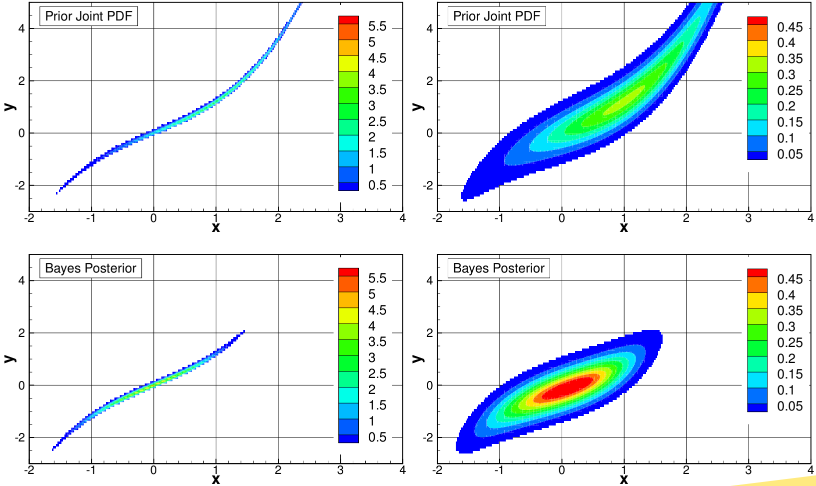
Nonlinear model including model errors



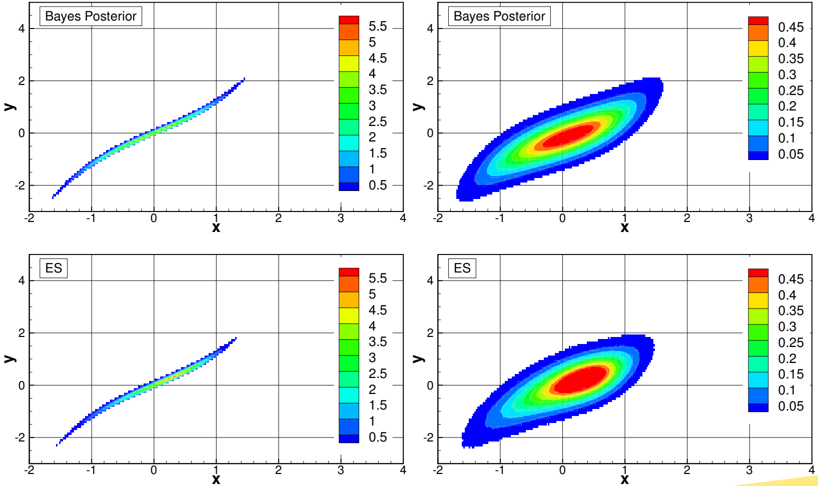
Pdfs for model error with nonlinear model



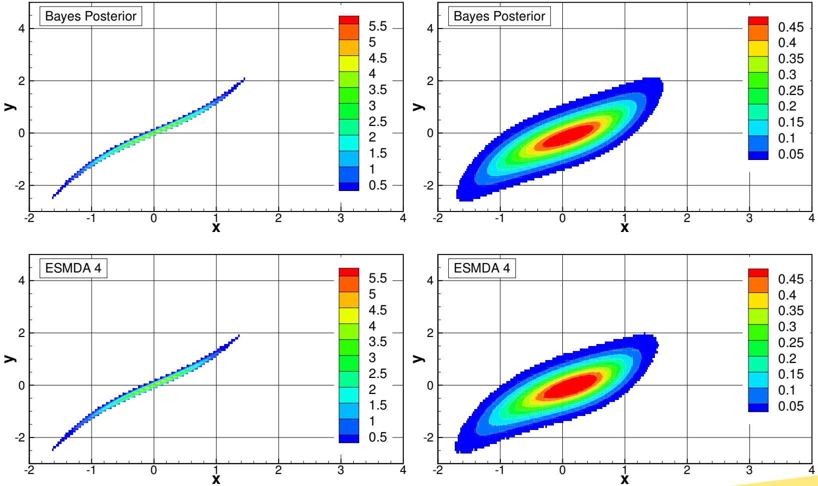
Joint pdfs nonlinear model



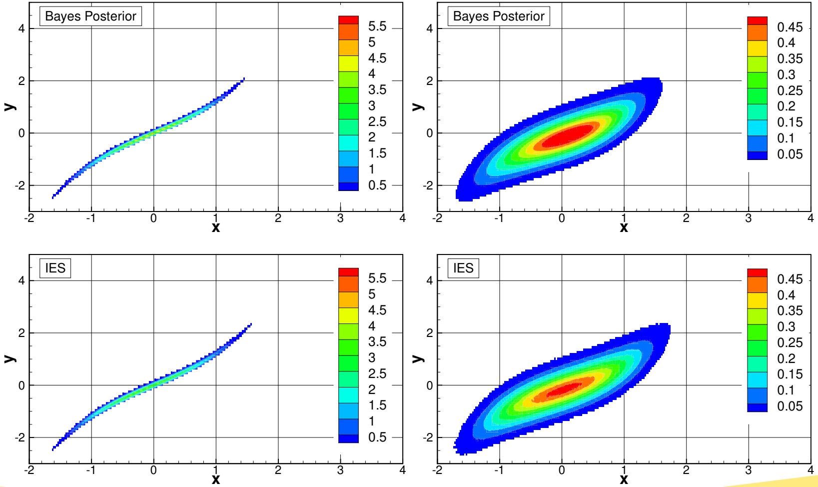
Joint pdfs nonlinear model



Joint pdfs nonlinear model



Joint pdfs nonlinear model



Acknowledgements

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Summary

IRIS

- We noted the duality of the DA and inverse problem.
- Iterations improve the estimate for models with modest nonlinearity.
- Iterative methods allows for inclusion of general model errors.
- Results published in Evensen (2018a,b).